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2002 J. Phys. A: Math. Gen. 35 9923

(http://iopscience.iop.org/0305-4470/35/46/314)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 9923–9930 PII: S0305-4470(02)38698-0

A transformation with symbolic computation and abundant new soliton-like solutions for the (1 + 2)-dimensional generalized Burgers equation

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Received 20 June 2002, in final form 26 September 2002
Published 7 November 2002
Online at stacks.iop.org/JPhysA/35/9923

Abstract
In this paper, an auto-Bäcklund transformation is presented for the generalized Burgers equation: $u_t + u_{xy} + \alpha uu_y + \alpha u_x\partial_x^{-1}u_y = 0$ (α is constant) by using an ansatz and symbolic computation. Particularly, this equation is transformed into a (1 + 2)-dimensional generalized heat equation $\omega_t + \omega_{xy} = 0$ by the Cole–Hopf transformation. This shows that this equation is C-integrable. Abundant types of new soliton-like solutions are obtained by virtue of the obtained transformation. These solutions contain n-soliton-like solutions, shock wave solutions and singular soliton-like solutions, which may be of important significance in explaining some physical phenomena. The approach can also be extended to other types of nonlinear partial differential equations in mathematical physics.

PACS numbers: 02.30.Ik, 05.45.Yv

1. Introduction

To date, many nonlinear soliton equations have been presented in nonlinear science [1, 2]. It is of important significance to study their properties, such as Painlevé integrability, C-integrability, Bäcklund transformation, Darboux transformation, exact soliton solutions and symmetries, etc. For example, the well-known Burgers equation [1–4]

$$u_t + uu_x - u_{xx} = 0 \tag{1}$$

has a remarkable property, that it can be transformed into a linear heat equation by using the famous Cole–Hopf transformation [3, 4]. In 1990, Webb and Zank [5] considered the Painlevé integrability of two higher-dimensional equations of the Burgers equation (1), i.e. the (1 + 2)-dimensional Burgers equation

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0 \tag{2}$$

and the (1 + 3)-dimensional Burgers equation

$$(u_t + uu_x - u_{xx})_x + u_{yy} + u_{zz} = 0. \tag{3}$$

Recently, we have [6] extended the idea of Tasso [7], which was used to deduce the Burgers hierarchy, to deduce two hierarchies of generalized Burgers equations in (2 + 1)-dimensional space. The simplest example is

$$u_t + u_{xy} + \alpha uu_y + \alpha u_x \partial_x^{-1} u_y = 0 \tag{4}$$

where $\partial_x = \partial/\partial_x$, $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$ and α is a constant, which was obtained from the Lax pair

$$\phi_x = u\phi \qquad \phi_t = V_n(u, u_x, u_y, \ldots)\phi$$

where $V_{n+1} = -\alpha u V_n - \partial_x V_n$, $V_1 = \partial_x^{-1} u_y$.

By using a series of transformations equation (4) reduces to the well-known Burgers equation. It is clear that equation (4) is different from equation (2). Thus, we call equation (4) the (1 + 2)-dimensional generalized Burgers equation, which describes wave propagation in (2 + 1)-dimensional space. The motion described by equation (4) is an isolated wave, localized in a small part of space. This will be shown in what follows. We [8] have given some dromion-like solutions of equation (4) when $\alpha = 1$. In this paper, we would like to obtain an auto-Bäcklund transformation (ABT) of equation (4) by using a new ansatz and symbolic computation. Particularly, equation (4) can be transformed into the (2 + 1)-dimensional heat equation by the Cole–Hopf transformation. As a consequence, many soliton-like solutions are found by virtue of the obtained transformation.

The rest of the paper is organized as follows. In section 2, an ABT of equation (4) is presented. In particular, equation (4) can be transformed into the (2 + 1)-dimensional heat equation by the Cole–Hopf transformation. In section 3, abundant explicit exact soliton-like solutions are obtained for equation (4) by some transformations with symbolic computation. Finally, some conclusions and some problems are given.

2. Auto-Bäcklund transformation of equation (4)

To deduce an ABT, we first introduce the following ideas:

(i) Hirota's dependent variable transformation [9] introduces a dependent variable $\phi(x, t)$ with a differentiator acting on its function $f(\phi(x, t)) = \ln(\phi(x, t))$.

(ii) the Clarkson–Kruskal method [10] considers a general function $F(x, t, \phi(x, t))$ and tries to establish an ordinary differential equation (ODE) for $\phi(x, t)$ so as to impose conditions upon F and ϕ.

(iii) Wang [11] has presented the homogeneous balance method to seek the solution of the given nonlinear partial differential equation (PDE)

$$u(x, y, t) = \partial_x^m \partial_y^s \{w[z(x, y, t)]\} + C. \tag{5}$$

(iv) Fan and Zhang [12] have extended the homogeneous balance method to search for the solution

$$u(x, t) = \partial_x^m \{w[z(x, t)]\} + u_0(x, t). \tag{6}$$

(v) The above-mentioned ideas have been extended to many nonlinear PDEs [13–17].

In the following, we would like to extend these ideas to equation (4). We assume that equation (4) possesses the following solution

$$u(x, y, t) = F(\partial_t, \partial_x, \partial_y)\{w[z(x, y, t)]\} + G(x, y, t) = \partial_x^m \partial_y^n f(w(x, y, t)) + u_0(x, y, t) \quad (7)$$

where m and n are parameters, and $u_0(x, y, t)$ is a function to be determined. According to the rule of leading-order analysis [11–17], balancing the highest-order linear term (i.e. u_{xy}) and nonlinear terms (i.e. uu_y or $u_x\partial_x^{-1}u_y$) leads to $m = 1$ and $n = 0$. Thus, equation (7) reduces to

$$u(x, y, t) = \partial_x f[w(x, y, t)] + u_0(x, y, t) = f'w_x + u_0 \quad (8)$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}w}$. With the aid of a symbolic computation package, Mathematica or Maple, from equation (8) we have

$$u_t = f''w_t w_x + f'w_{xt} + u_{0t} \quad (9)$$

$$u_{xy} = f'''w_x^2 w_y + f''w_{xx}w_y + 2f''w_x w_{xy} + f'w_{xxy} + u_{0xx} \quad (10)$$

$$uu_y = f'f''w_x^2 w_y + f'^2 w_x w_{xy} + f''u_0 w_x w_y + f'u_{0y}w_x + f'u_0 w_{xy} + u_0 u_{0y} \quad (11)$$

$$u_x\partial_x^{-1}u_y = f'f''w_x^2 w_y + f'^2 w_{xx}w_y + f''w_x^2\partial_x^{-1}u_{0y} + f'w_y u_{0x} + f'w_{xx}\partial_x^{-1}u_{0y}. \quad (12)$$

Substituting equations (9)–(12) into equation (4) and collecting all homogeneous terms with respect to partial derivatives of $w(x, y, t)$, we have

$$\begin{aligned}(f''' + 2\alpha f'f'')w_x^2 w_y &+ \big[f''w_t v_x + f''w_{xx}v_y + 2f''w_x w_{xy} + \alpha\big(f'^2 w_x w_{xy} + f''u_0 w_x w_y \\ &+ f'^2 w_{xx}w_y + f''w_x^2\partial_x^{-1}u_{0y}\big)\big] + \big[w_{xy} + w_{xxy} + \alpha\big(u_{0y}w_x + u_0 w_{xy} + w_y u_{0x} \\ &+ w_{xx}\partial_x^{-1}u_{0y}\big)\big]f' + u_{0t} + u_{0xy} + \alpha u_0 u_{0y} + \alpha u_{0x}\partial_x^{-1}u_{0y} = 0. \end{aligned} \quad (13)$$

To fix the unknown function $f(w)$, setting the coefficient of $w_x^2 w_y$ in equation (13) to zero yields a nonlinear third-order ODE of f

$$f''' + 2\alpha f'f'' = 0 \quad (14)$$

which indicates that the nonlinear terms (i.e. uu_y and $u_x\partial_x^{-1}u_y$) and the highest-order linear term (i.e. u_{xy}) have been partially balanced. It is easy to show that equation (14) has the solution

$$f(w(x, y, t)) = \frac{1}{\alpha}\ln w(x, y, t). \quad (15)$$

From equation (15), we easily have the following relation

$$f'^2 = -\frac{1}{\alpha}f''. \quad (16)$$

By using equations (14) and (16), equation (13) becomes an equation involving f'', f' and $f^0 = 1$. According to the linear independence of f'', f' and $f^0 = 1$, we have the following system of over-determined PDEs with respect to the variables $w(x, y, t)$ and $u_0(x, y, t)$:

$$w_x\big(w_t + w_{xy} + \alpha u_0 w_y + \alpha w_x\partial_x^{-1}u_{0y}\big) = 0 \quad (17)$$

$$\partial_x\big(w_t + w_{xy} + \alpha u_0 w_y + \alpha w_x\partial_x^{-1}u_{0y}\big) = 0 \quad (18)$$

$$u_{0t} + u_{0xy} + \alpha u_0 u_{0y} + \alpha u_{0x}\partial_x^{-1}u_{0y} = 0. \quad (19)$$

From equation (19), it is clear that u_0 is just a solution of equation (1). Therefore, the substitution of equation (15) into equation (8) leads to an ABT for equation (4)

$$u(x, y, t) = \frac{1}{\alpha}\frac{\partial}{\partial x}[\ln w(x, y, t) + u_0(x, y, t)] = \frac{1}{\alpha}\frac{w_x}{w} + u_0(x, y, t) \quad (20)$$

where w and u_0 satisfy equations (17)–(19).

Remark. If we set $u_0 = 0$ in the ABT (20), then the ABT (20) becomes the famous Cole–Hopf transformation of equation (4)

$$u(x, y, t) = \frac{1}{\alpha} \frac{w_x}{w}. \tag{21}$$

Under the transformation, a bilinear equation of equation (1) is given by

$$w_{xt} w - w_x w_t + w_{xxy} w - w_x w_{xy} = 0 \tag{22}$$

which can be simply reduced to the (2 + 1)-dimensional heat equation $\omega_t + \omega_{xy} = 0$ by using the Cole–Hopf transformation (21).

3. Explicit and exact soliton-like solutions

By virtue of the ABT (20), we reduce equation (4) to a system of over-determined PDEs (17)–(19). If we can obtain the solution (u_0, w) of equations (17)–(19), then a solution u of equation (4) is deduced from the ABT (20). Now we mainly consider equations (17)–(19). We can see that the simple trial solution for the system of equations (17)–(19),

$$w(x, y, t) = 1 + \exp(kx + \beta y + \gamma t + \delta) \tag{23}$$

would lead to nothing but shock waves, where k, β, γ and δ are constants.

Nevertheless, this clue inspires us to proceed further and to find solutions other than solitary waves as follows:

Case 1. We propose that equations (17)–(19) have the following *x*-linear formal solution with $u_0 =$ constant, which is a solution of equation (4)

$$w(x, y, t) = P(y, t) + \sum_{i=1}^{N} \mu_i \exp[\Theta_i(y, t)x + \Psi_i(y, t)] \tag{24}$$

where $\mu_i = \pm 1$ and $P_i(y, t)$, $\Theta_i(y, t)$ and $\Psi_i(y, t)$ are differentiable functions of y and t only to be determined later.

Substituting equation (24) into equations (17)–(19) with symbolic computation, we have

$$\begin{cases} P_t + \alpha u_0 P_y = 0 \\ \Theta_{it} + \Theta_i \Theta_{iy} + \alpha u_0 \Theta_{iy} = 0 \\ \Psi_{it} + \Theta_{iy} + \Theta_i \Psi_{iy} + \alpha u_0 \Psi_{iy} = 0. \end{cases} \tag{25}$$

Therefore, we have the N-soliton-like solution of equation (4) from equations (20), (24) and (25)

$$u = \frac{1}{\alpha} \frac{\sum_{i=1}^{N} \mu_i \Theta_i(y, t) \exp[\Theta_i(y, t)x + \Psi_i(y, t)]}{P(y, t) + \sum_{i=1}^{N} \mu_i \exp[\Theta_i(y, t)x + \Psi_i(y, t)]} + u_0 \tag{26}$$

where $P(y, t)$, $\Theta_i(y, t)$ and $\Psi_i(y, t)$ satisfy (25).

In the following, we would like to give some special solutions of equation (25) so that the corresponding solutions of (4) are found.

Solution 1. We have the solutions from equation (25)

$$P(y, t) = p(y - \alpha u_0 t) \qquad \Theta_i(y, t) = \frac{y - y_i}{t - t_i} - \alpha u_0 \qquad \Psi_i(y, t) = \ln \frac{(t - t_i)^{\beta_i}}{(y - y_i)^{1+\beta_i}} \tag{27}$$

where $p(y - \alpha u_0 t)$ is an arbitrary function of $(y - \alpha u_0 t)$, and y_i, t_i and β_i are arbitrary constants.

Therefore, we have the n-soliton-like solution of equation (4) from equations (20), (24) and (27):

$$u_1 = \frac{1}{\alpha}\frac{\sum_{i=1}^{N}\mu_i\left(\frac{y-y_i}{t-t_i}-\alpha u_0\right)\exp\left[\left(\frac{y-y_i}{t-t_i}-\alpha u_0\right)x+\ln\frac{(t-t_i)^{\beta_i}}{(y-y_i)^{1+\beta_i}}\right]}{p(y-\alpha u_0 t)+\sum_{i=1}^{N}\mu_i\exp\left[\left(\frac{y-y_i}{t-t_i}-\alpha u_0\right)x+\ln\frac{(t-t_i)^{\beta_i}}{(y-y_i)^{1+\beta_i}}\right]}+u_0. \tag{28}$$

In particular, when $N = 1$ we have:

Family A (Shock-like wave solution)**.** When $p(y-\alpha u_0 t) > 0$, $\mu_1 = 1$,

$$u_{11} = \frac{1}{2\alpha}\left(\frac{y-y_1}{t-t_1}-\alpha u_0\right)\left\{1+\tanh\frac{1}{2}\left[\left(\frac{y-y_1}{t-t_1}-\alpha u_0\right)x + \ln\frac{(t-t_0)^{\beta}}{(y-y_1)^{1+\beta}}-\ln p(y-\alpha u_0 t)\right]\right\}+u_0. \tag{29a}$$

Family B (Singular soliton-like solution)**.** When $p(y-\alpha u_0 t) < 0$, $\mu_1 = 1$,

$$u_{12} = \frac{1}{2\alpha}\left(\frac{y-y_1}{t-t_1}-\alpha u_0\right)\left\{1+\coth\frac{1}{2}\left[\left(\frac{y-y_1}{t-t_1}-\alpha u_0\right)x + \ln\frac{(t-t_0)^{\beta}}{(y-y_1)^{1+\beta}}-\ln|p(y-\alpha u_0 t)|\right]\right\}+u_0. \tag{29b}$$

Solution 2. We have another family of solutions from (25)

$$P(y,t) = p(y-\alpha u_0 t) \qquad \Theta_i(y,t) = \theta_i = \text{const} \qquad \Psi_i(y,t) = \psi_i(y-(\theta_i+\alpha u_0)t) \tag{30}$$

where $p(y-\alpha u_0 t)$ and $\psi_i(y-(\theta_i+\alpha u_0)t)$ are arbitrary functions of $(y-\alpha u_0 t)$ and $(y-(\theta_i+\alpha u_0)t)$ respectively, and θ_i is an arbitrary constant.

Therefore, we have another n-soliton-like solution of equation (4) from equations (20), (24) and (30)

$$u_2 = \frac{1}{\alpha}\frac{\sum_{i=1}^{N}\mu_i\theta_i\exp[\theta_i x+\psi_i(y-(\theta_i+\alpha u_0)t)]}{p(y-\alpha u_0 t)+\sum_{i=1}^{N}\mu_i\exp[\theta_i x+\psi_i(y-(\theta_i+\alpha u_0)t)]}+u_0. \tag{31}$$

In particular, when $N = 1$ we have:

Family C (Shock-like wave solution)**.** When $p(y-\alpha u_0 t) > 0$, $\mu_1 = 1$,

$$u_{21} = \frac{\theta_1}{2\alpha}\left\{1+\tanh\frac{1}{2}[\theta_1 x+\phi_1(y-(\theta_1+\alpha u_0)t)-\ln p(y-\alpha u_0 t)]\right\}+u_0. \tag{32a}$$

Family D (Singular soliton-like solution)**.** When $p(y-\alpha u_0 t) < 0$, $\mu_1 = 1$,

$$u_{22} = \frac{\theta_1}{2\alpha}\left\{1+\coth\frac{1}{2}[\theta_1 x+\phi_1(y-(\theta_1+\alpha u_0)t)-\ln|p(y-\alpha u_0 t)|]\right\}+u_0. \tag{32b}$$

Case 2. We assume that equations (17)–(19) have the following y-linear formal solution with $u_0 = u_0(x)$ which is a solution of equation (4)

$$w(x,y,t) = P(x,t)+\sum_{j=1}^{N}\mu_j\exp[\Theta_j(x,t)y+\Psi_j(x,t)] \tag{33}$$

where $\mu_j = \pm 1$, and $P(x,t)$, $\Theta_j(x,t)$ and $\Psi_j(x,t)$ are differentiable functions of x and t to be determined later.

Substituting equation (33) into equations (17)–(19) with symbolic computation, we have

$$\begin{cases} P_t = 0 \quad \text{i.e. } P(x,t) = p(x) \\ \Theta_{jt} + \Theta_j \Theta_{jx} = 0 \\ \Psi_{jt} + \Theta_{jx} + \Theta_j \Psi_{jx} + \alpha u_0(x) \Theta_j = 0 \end{cases} \tag{34}$$

where $p(x)$ is an arbitrary function of x.

Therefore, we have the n-soliton-like solution from equations (20), (33) and (34)

$$u = \frac{1}{\alpha} \frac{p'(x) + \sum_{j=1}^{N} \mu_j [\Theta_{jx}(x,t)y + \Psi_{jx}(x,t)] \exp[\Theta_j(x,t)y + \Psi_j(x,t)]}{p(x) + \sum_{j=1}^{N} \mu_j \exp[\Theta_j(x,t)x + \Psi_j(x,t)]} + u_0(x) \tag{35}$$

where $P(y,t)$, $\Theta_j(y,t)$ and $\Psi_j(y,t)$ satisfy equation (34).

In the following, we would like to give some special solutions of equation (34) so that the corresponding solutions of equation (4) are found.

Solution 3. We have the solutions from equation (34)

$$P(x,t) = p(x) \qquad \Theta_j(x,t) = \frac{x - x_j}{t - t_j} \qquad \Psi_j(x,t) = \ln \frac{(t - t_j)^{\gamma_j}}{(x - x_j)^{1+\gamma_j}} - \alpha \int^x u_0(s)\,\mathrm{d}s \tag{36}$$

where $p(x)$ is an arbitrary function of x, and x_j, t_j and γ_j are arbitrary constants.

Therefore, we have the n-soliton-like solution of equation (4) from equations (20), (35) and (36)

$$u_3 = \frac{1}{\alpha} \frac{p'(x) + \sum_{j=1}^{N} \mu_j \left(\frac{y}{t-t_j} - \frac{1+\gamma_j}{x-x_j} - \alpha u_0(x)\right) \exp\left[\frac{x-x_j}{t-t_j} y + \ln \frac{(t-t_j)^{\gamma_j}}{(x-x_j)^{1+\gamma_j}} - \alpha \int^x u_0(s)\,\mathrm{d}s\right]}{p(x) + \sum_{j=1}^{N} \mu_j \exp\left[\frac{x-x_j}{t-t_j} y + \ln \frac{(t-t_j)^{\gamma_j}}{(x-x_j)^{1+\gamma_j}} - \alpha \int^x u_0(s)\,\mathrm{d}s\right]} + u_0(x). \tag{37}$$

In particular when $N = 1$ we have:

Family E (Shock-like wave solution)**.** When $p(x) > 0$, $\mu_1 = 1$,

$$\begin{aligned} u_{31} = \frac{1}{2\alpha} &\left[-\frac{p'(x)}{p(x)} + \frac{y}{t - t_1} - \frac{1+\gamma_1}{x - x_1} - \alpha u_0(x)\right] \\ &\times \tanh \frac{1}{2}\left[\frac{x - x_1}{t - t_1} y + \ln \frac{(t - t_1)^{\gamma_1}}{(x - x_1)^{1+\gamma_1}} - \alpha \int^x u_0(s)\,\mathrm{d}s - \ln p(x)\right] \\ &+ \frac{1}{2\alpha}\left[\frac{p'(x)}{p(x)} - \frac{y}{t - t_1} + \frac{1+\gamma_1}{x - x_1} - \alpha u_0(x)\right] + u_0(x). \end{aligned} \tag{38a}$$

Family F (Soliton-like solution)**.** When $p(x) < 0$, $\mu_1 = 1$,

$$\begin{aligned} u_{32} = \frac{1}{2\alpha} &\left[-\frac{p'(x)}{p(x)} + \frac{y}{t - t_1} - \frac{1+\gamma_1}{x - x_1} - \alpha u_0(x)\right] \\ &\times \coth \frac{1}{2}\left[\frac{x - x_1}{t - t_1} y + \ln \frac{(t - t_1)^{\gamma_1}}{(x - x_1)^{1+\gamma_1}} - \alpha \int^x u_0(s)\,\mathrm{d}s - \ln |p(x)|\right] \\ &+ \frac{1}{2\alpha}\left[\frac{p'(x)}{p(x)} - \frac{y}{t - t_1} + \frac{1+\gamma_1}{x - x_1} - \alpha u_0(x)\right] + u_0(x). \end{aligned} \tag{38b}$$

Solution 4. We have another family of solutions from equation (34)

$$P(x,t) = p(x) \qquad \Theta_j(x,t) = \theta_j = \text{const} \qquad \Psi_j(x,t) = -\alpha \int^x u_0(s)\,\mathrm{d}s - \frac{\lambda_j}{\theta_j}x + \lambda_j t \tag{39}$$

where $p(x)$ is an arbitrary function of x, and θ_j and λ_j are arbitrary constants.

Therefore, we have the n-soliton-like solution of equation (4) from equations (20), (35) and (39)

$$u_4 = \frac{1}{\alpha} \frac{p'(x) + \sum_{j=1}^N \mu_j\left(-\frac{\lambda_j}{\theta_j} - \alpha u_0(x)\right) \exp\left[\theta_j y - \alpha \int^x u_0(s)\,\mathrm{d}s - \frac{\lambda_j}{\theta_j}x + \lambda_j t\right]}{p(x) + \exp\left[\theta_j y - \alpha \int^x u_0(s)\,\mathrm{d}s - \frac{\lambda_j}{\theta_j}x + \lambda_j t\right]} + u_0(x). \tag{40}$$

In particular, when $N = 1$ we have:

Family G (Shock-like wave solution)**.** When $p(x) > 0$, $\mu_1 = 1$

$$u_{41} = -\frac{1}{2\alpha}\left[\frac{p'(x)}{p(x)} + \alpha u_0(x) + \frac{\lambda_1}{\theta_1}\right] \tanh \frac{1}{2}\left[\theta_1 y - \alpha \int^x u_0(s)\,\mathrm{d}s - \frac{\lambda_1}{\theta_1}x + \lambda_1 t - \ln p(x)\right] + \frac{1}{2\alpha}\left[\frac{p'(x)}{p(x)} - \alpha u_0(x) - \frac{\lambda_1}{\theta_1}\right] + u_0(x). \tag{41a}$$

Family H (Soliton-like wave solution)**.** When $p(x) < 0$, $\mu_1 = 1$

$$u_{42} = -\frac{1}{2\alpha}\left[\frac{p'(x)}{p(x)} + \alpha u_0(x) + \frac{\lambda_1}{\theta_1}\right] \coth \frac{1}{2}\left[\theta_1 y - \alpha \int^x u_0(s)\,\mathrm{d}s - \frac{\lambda_1}{\theta_1}x + \lambda_1 t - \ln |p(x)|\right] + \frac{1}{2\alpha}\left[\frac{p'(x)}{p(x)} - \alpha u_0(x) - \frac{\lambda_1}{\theta_1}\right] + u_0(x). \tag{41b}$$

Case 3. When $u_0 =$ constant, we know that equations (17)–(19) have the solution

$$w(x,y,t) = a_0 + \sum_{j=1}^N a_j \exp[k_j x + l_j y - (k_j l_j + l_j \alpha u_0)t + c] \tag{42}$$

where c, k_j, l_j and a_j are arbitrary constants.

Thus, we obtain the n-soliton-like solution of equation (4) from equations (20) and (42)

$$u(x,y,t) = \frac{1}{\alpha} \frac{\sum_{j=1}^N a_j k_j \exp[k_j x + l_j y - (k_j l_j + l_j \alpha u_0)t + c]}{a_0 + \sum_{j=1}^N a_j \exp[k_j x + l_j y - (k_j l_j + l_j \alpha u_0)t + c]} + u_0. \tag{43}$$

Therefore, from equation (43) we obtain:

Family I (Shock wave solution)**.** When $N = 1, a_0 a_1 > 0$,

$$u_1 = \frac{k_1}{2\alpha}\left[1 + \tanh \frac{k_1 x + l_1 y - (k_1 l_1 + l_1 \alpha u_0)t - \ln \frac{a_1}{a_0} + c}{2}\right] + u_0 \tag{44a}$$

which is an isolated wave, localized in a small part of space.

Family J (Singular soliton-like solution)**.** When $N = 1$, $a_0a_1 < 0$,

$$u_2 = \frac{k_1}{2\alpha}\left[1 + \coth\frac{k_1x + l_1y - (k_1l_1 + l_1\alpha u_0)t - \ln\left(-\frac{a_1}{a_0}\right) + c}{2}\right] + u_0 \qquad (44b)$$

which is a singular soliton-like solution and shows that the solution blows up at the point (x_0, y_0) for a certain time $t = t_0$.

Family K (Double soliton-like solutions)**.** When $N = 2$,

$$u = \frac{1}{\alpha}\frac{a_1k_1\,\mathrm{e}^{k_1x+l_1y-(k_1l_1+l_1\alpha u_0)t+c} + a_2k_2\,\mathrm{e}^{k_2x+l_2y-(k_2l_2+l_2\alpha u_0)t+c}}{a_0 + a_1\,\mathrm{e}^{k_1x+l_1y-(k_1l_1+l_1\alpha u_0)t+c}a_2\,\mathrm{e}^{k_2x+l_2y-(k_2l_2+l_2\alpha u_0)t+c}} + u_0 \qquad (45)$$

where a_i and k_i, l_i $(i = 1, 2)$ are arbitrary constants.

In summary, by using symbolic computation and some simple transformations, we have investigated many families of explicit and exact solutions of (1 + 2)-dimensional generalized Burgers equations, which include n-soliton-like solutions, shock-like wave solutions and singular soliton-like solutions. These solutions may be of great significance in explaining some physical phenomena. The approach can also be extended to other nonlinear evolution equations in mathematical physics. Further study is needed to see whether there are other types of exact solutions.

Acknowledgments

The author thanks the anonymous referee for her/his valuable suggestions. The project was in part supported by the PIMS post-doctoral fellowship and NSERC.

References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[2] Gu C H *et al* 1995 *Soliton Theory and its Applications* (Berlin: Springer)
[3] Cole J D 1951 *Q. Appl. Math.* **9** 225
[4] Hopf E 1950 *Commun. Pure Appl. Math.* **3** 201
[5] Webb G M and Zank G P 1990 *J. Phys. A: Math. Gen.* **23** 5465
Webb G M and Zank G P 1990 *Phys. Lett.* A **150** 14
[6] Yan Z Y unpublished
[7] Tasso H 1996 *J. Phys. A: Math. Gen.* **29** 7779
[8] Yan Z Y 2002 *Chin. J. Phys.* **40** 203
[9] Hirota R 1979 *J. Phys. Soc. Japan* **46** 312
[10] Nucci M and Clarkson P A 1992 *Phys. Lett.* A **164** 49
[11] Wang M L 1995 *Phys. Lett.* A **199** 169
[12] Fan E and Zhang H 1998 *Appl. Math. Mech.* **19** 645
[13] Yan Z Y and Zhang H Q 2001 *J. Phys. A: Math. Gen.* **34** 1785
[14] Yan Z Y 2001 *Commun. Theor. Phys.* **36** 135
[15] Yan Z Y and Zhang H Q 2002 *Comput. Math. Appl.* at press
[16] Yan Z Y and Zhang H Q 1999 *Commun. Nonlinear Sci. Numer. Simul.* **4** 146
[17] Tian B and Gao Y T 1996 *J. Phys. A: Math. Gen.* **29** 2895